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MÖBIUS INVARIANT HILBERT SPACES OF HOLOMORPHIC FUNCTIONS IN THE UNIT BALL OF $\mathbb{C}^n$

KEHE ZHU

ABSTRACT. We prove that there exists a unique Hilbert space of holomorphic functions in the open unit ball of $\mathbb{C}^n$ whose (semi-) inner product is invariant under Möbius transformations.

1. INTRODUCTION

Let B_n be the open unit ball in $\mathbb{C}^n$ and $\operatorname{Aut}(B_n)$ be the Möbius group of biholomorphic mappings from B_n onto B_n. Let H be a Hilbert space of holomorphic functions in B_n. In this paper, a Hilbert space will be a linear space with a complete semi-inner product. Furthermore, we assume that H contains all polynomials and the polynomials are dense in H. We say that H is Möbius invariant if $f \circ \varphi \in H$ and $\|f \circ \varphi\| = \|f\|$ whenever $f \in H$ and $\varphi \in \operatorname{Aut}(B_n)$. The main result of the paper is the following

Theorem. *There exists a unique Hilbert space of holomorphic functions in B_n which is Möbius invariant.*

When $n = 1$, the above result was established in [1]. The unique Möbius invariant Hilbert space in this case is the Dirichlet space $\mathscr{D}$ consisting of holomorphic functions f in the open unit disc $\mathbb{D}$ of $\mathbb{C}$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < +\infty,$$

where dA is the (normalized) area measure on $\mathbb{D}$. The (semi-) inner product in $\mathscr{D}$ is given by

$$\langle f, g\rangle = \int_{\mathbb{D}} f'(z)\overline{g'(z)} \, dA(z).$$

We remark that the condition $\|f \circ \varphi\| = \|f\|$ for all $f \in H$ and $\varphi \in \operatorname{Aut}(B_n)$ is equivalent to the condition $\langle f \circ \varphi, g \circ \varphi\rangle = \langle f, g\rangle$ for all $f, g \in H$ and $\varphi \in \operatorname{Aut}(B_n)$.

The organization of the paper is as follows: In the next section, we present some preliminary results which will be needed for the proof of the main theorem. In §3, we prove the uniqueness of invariant (semi-) inner products. In

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§4 we prove that the inner product obtained in the uniqueness proof is indeed Möbius invariant. This will establish the existence. §§5 and 6 discuss other possible ways of describing the invariant inner product and the invariant Hilbert space.

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2. Preliminary results

For any ordered n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, we use the following abbreviated notations:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$
$$\alpha! = \alpha_1! \cdots \alpha_n!,$$
$$z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$
$$\frac{\partial^{|\alpha|} f}{\partial z^\alpha} = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}},$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $f(z)$ is holomorphic in B_n. Recall that the inner product in $\mathbb{C}^n$ is given by

$$\langle z, w \rangle = \sum_{k=1}^{n} z_k \bar{w}_k.$$

We simply write

$$|z| = \sqrt{\langle z, z \rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$$

for any z in $\mathbb{C}^n$.

Let $\mathscr{U}_n$ be the group of unitary operators on the Hilbert space $\mathbb{C}^n$. It is clear that $\mathscr{U}_n$ is a subgroup of $\mathrm{Aut}(B_n)$. In fact, $\mathscr{U}_n$ is the isotropy subgroup of $\mathrm{Aut}(B_n)$ at 0 when we consider the natural action of $\mathrm{Aut}(B_n)$ on B_n. Therefore, for $\varphi \in \mathrm{Aut}(B_n)$, we have $\varphi \in \mathscr{U}_n$ iff $\varphi(0) = 0$.

For any $a \in B_n$, define $\varphi_a \in \mathrm{Aut}(B_n)$ as follows: If $a = 0$, then $\varphi_a(z) \equiv -z$. If $a \neq 0$, then

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - |a|^2} P_a^\perp z}{1 - \langle z, a \rangle},$$

where P_a is the orthogonal projection from $\mathbb{C}^n$ onto the complex line $[a]$ spanned in $\mathbb{C}^n$ by a, $P_a^\perp$ is the orthogonal projection to the orthogonal compliment of $[a]$ in $\mathbb{C}^n$. It is easy to see that $\varphi_a(0) = a$, $\varphi_a(a) = 0$, and $\varphi_a \circ \varphi_a(z) \equiv z$. For these and other properties of φ_a, see [6, 7, 4].

Given $\varphi \in \mathrm{Aut}(B_n)$, let $a = \varphi^{-1}(0)$ and $U = \varphi \circ \varphi_a$, then $U \in \mathrm{Aut}(B_n)$ and $U(0) = \varphi(\varphi_a(0)) = \varphi(a) = 0$. Thus U is a unitary. Since φ_a is involutive, we have $\varphi = U \circ \varphi_a$. This shows that $\mathrm{Aut}(B_n)$ is generated by $\mathscr{U}_n$ and $\{\varphi_a : a \in B_n\}$. The following lemma will be needed in the proof of the main theorem.

Lemma 1. $\mathrm{Aut}(B_n)$ *is generated by* $\mathscr{U}_n$ *and all* φ_a *with* $a = (r, 0, \ldots, 0)$ *and* $0 \le r < 1$.

Proof. When $a = (r, 0, \ldots, 0)$, we simply write $\varphi_a = \varphi_r$. Since $\mathrm{Aut}(B_n)$ is generated by $\mathscr{U}_n$ and $\{\varphi_a : a \in B_n\}$, it suffices to show that each φ_a $(a \in B_n)$ can be written as a product of unitaries and some φ_r $(0 \le r < 1)$.

Given $a \in B_n$. If $a = 0$, then φ_a is already a unitary. So we may assume $a \ne 0$. We show that there are $U, V \in \mathscr{U}_n$ such that $\varphi_a = U\varphi_{|a|}V$. Suppose $U \in \mathscr{U}_n$, then UP_aU^* is a one-dimensional projection in $\mathbb{C}^n$. Since

$$UP_aU^*(Ua) = UP_a(a) = Ua,$$

UP_aU^* must be the projection onto the complex line $[Ua]$. Thus $UP_aU^* = P_{Ua}$. It follows that $UP_a^{\perp}U^* = P_{Ua}^{\perp}$ and $U\varphi_aU^* = \varphi_{Ua}$.

It is easy to see that there exists a unitary $U \in \mathscr{U}_n$ such that $U(|a|, 0, \ldots, 0) = a$. The above argument now implies that $\varphi_a = U\varphi_{|a|}U^*$, completing the proof of Lemma 1. □

Note that by the definition of φ_a, we have

$$\varphi_r(z) = \left(\frac{r - z_1}{1 - rz_1}, -\frac{\sqrt{1-r^2}\,z_2}{1 - rz_1}, \ldots, -\frac{\sqrt{1-r^2}\,z_n}{1 - rz_1}\right)$$

for all $z \in B_n$ and $r \in [0, 1)$.

In this paper, we'll be assuming that all Hilbert spaces of holomorphic functions in B_n contain the polynomials and the polynomials are dense in them. It is natural to ask if this condition is always satisfied. For invariant function spaces, we have

Lemma 2. *Suppose* H *is a linear space of holomorphic functions in* B_n *with a complete semi-inner product which is invariant under* $\mathrm{Aut}(B_n)$. *Then* H *contains all the polynomials and the polynomials are dense in it iff* H *contains a nonconstant function and* $\mathscr{U}_n$ *acts on* H *continuously.*

Proof. The "only if" part will follow from our main result. The "if" part has a proof similar to that of the one-dimensional case. See Proposition 2 in [2] and Lemma 3 in [8]. □

Remark. We will not need Lemma 2 in the proof of the main theorem.

Let $dm_n = dx_1\,dy_1 \ldots dx_n\,dy_n$ be the Euclidean measure in $\mathbb{C}^n = R^{2n}$. For any $r > 0$, the volume of any Euclidean ball in $\mathbb{C}^n$ with radius r is $\pi^n r^{2n}/n!$. Let $dV_n = \frac{n!}{\pi^n}\,dm_n$ be the normalized volume measure on B_n so that $\int_{B_n} dV_n(z) = 1$. When $n = 1$, we write $dA = dV_1$. Clearly, $dA = \frac{1}{\pi}\,dx\,dy = \frac{1}{\pi} r\,dr\,d\theta$ in polar coordinates. The following lemma will be needed in our later discussions.

Lemma 3. *If* $n \ge 1$ *and* $f(z) = f(z_1)$ *only depends on* z_1, *then*

$$\int_{B_n} f(z)\,dV_n(z) = n\int_{\mathbb{D}} (1 - |z_1|^2)^{n-1} f(z_1)\,dA(z_1),$$

where $\mathbb{D} = B_1$ *is the open unit disc in* $\mathbb{C}$.

Proof.

$$\begin{aligned}\int_{B_n} f(z)\,dV_n(z) &= \frac{n!}{\pi^n}\int_{B_n} f(z)\,dm_n(z)\\ &= \frac{n!}{\pi^n}\int_{\mathbb{D}} f(z_1)\,dx_1\,dy_1 \int_{|z_2|^2+\cdots+|z_n|^2<1-|z_1|^2} dx_2\,dy_2\cdots dx_n\,dy_n\\ &= \frac{n!}{\pi^n}\int_{\mathbb{D}} f(z_1)\frac{\pi^{n-1}(1-|z_1|^2)^{n-1}}{(n-1)!}\,dx_1\,dy_1\\ &= n\int_{\mathbb{D}}(1-|z_1|^2)^{n-1} f(z_1)\frac{dx_1\,dy_1}{\pi}\\ &= n\int_{\mathbb{D}}(1-|z_1|^2)^{n-1} f(z_1)\,dA(z_1). \quad \square\end{aligned}$$

3. The uniqueness

Suppose H is a Hilbert space of holomorphic functions in B_n with (semi-) inner product $\langle\ ,\ \rangle$. For any $f, g \in H$, we write

$$f(z) = \sum_\alpha a_\alpha z^\alpha, \qquad g(z) = \sum_\beta b_\beta z^\beta .$$

Since the polynomials are dense in H, we have

$$\langle f, g\rangle = \sum_{\alpha,\beta} a_\alpha \bar{b}_\beta \langle z^\alpha, z^\beta\rangle .$$

We compute the inner product $\langle z^\alpha, z^\beta\rangle$ in this section under the assumption that $\langle\ ,\ \rangle$ be Möbius invariant, that is, $\langle f\circ\varphi, g\circ\varphi\rangle = \langle f, g\rangle$ if $f, g \in H$ and $\varphi \in \mathrm{Aut}(B_n)$. The main result of this section is

Theorem 4. *If H is a Hilbert space of holomorphic functions in B_n with a nonzero Möbius invariant (semi-) inner product $\langle\ ,\ \rangle$, then*

$$\langle f, g\rangle = c\sum_\alpha a_\alpha \bar{b}_\alpha \frac{\alpha!}{|\alpha|!}|\alpha|$$

for some constant $c > 0$ and all $f(z) = \sum_\alpha a_\alpha z^\alpha$, $g(z) = \sum_\alpha b_\alpha z^\alpha$ in H.

It follows from the above theorem that if H is an invariant Hilbert space of holomorphic functions in B_n, then

$$H = \left\{ f(z) = \sum_\alpha a_\alpha z^\alpha :\ \sum_\alpha |a_\alpha|^2 \frac{\alpha!}{|\alpha|!}|\alpha| < +\infty \right\} .$$

Thus H is unique. Moreover, the above theorem also implies that the inner product in H is unique up to a positive multiple. The canonical inner product in H is

$$\langle f, g\rangle = \sum_\alpha a_\alpha \bar{b}_\alpha \frac{\alpha!}{|\alpha|!}|\alpha| .$$

It is not clear at all that the above inner product is Möbius invariant. This will be proved in the next secton, thus establishing the existence of invariant Hilbert spaces of holomorphic functions in B_n.

In order to prove Theorem 4, it suffices to prove the following three equalities:

$$\langle z^\alpha , z^\beta \rangle = 0 \quad \text{if } \alpha \neq \beta , \tag{1}$$

$$\langle z^\alpha , z^\alpha \rangle = \frac{\alpha!}{|\alpha|!} \langle z_1^{|\alpha|} , z_1^{|\alpha|} \rangle , \tag{2}$$

$$\langle z_1^k , z_1^k \rangle = k \langle z_1 , z_1 \rangle . \tag{3}$$

The constant c in Theorem 4 is then $\langle z_1 , z_1 \rangle$.

The proof of (1) is almost trivial. Assume that $\alpha \neq \beta$, then there exists some $1 \leq k \leq n$ such that $\alpha_k \neq \beta_k$. We may as well assume $\alpha_1 \neq \beta_1$. Let U be the unitary operator on $\mathbb{C}^n$ defined by

$$U(z_1 , z_2 , \dots , z_n) = (z_1 e^{i\theta} , z_2 , \dots , z_n),$$

where θ is any real number. Since H is invariant under $\mathscr{U}_n$, we must have

$$\langle z^\alpha , z^\beta \rangle = \langle z^\alpha \circ U , z^\beta \circ U \rangle = e^{i(\alpha_1 - \beta_1)\theta} \langle z^\alpha , z^\beta \rangle .$$

Since θ is arbitrary and $\alpha_1 \neq \beta_1$, we must have $\langle z^\alpha , z^\beta \rangle = 0$.

The proof of (3) is similar to that of Theorem 1 in [1]. We reproduce it here for completeness.

Recall that for $r \in [0, 1)$,

$$\varphi_r(z) = \left(\frac{r - z_1}{1 - r z_1} , - \frac{\sqrt{1 - r^2} z_2}{1 - r z_1} , \dots , - \frac{\sqrt{1 - r^2} z_n}{1 - r z_1} \right) .$$

The invariance of $\langle \ , \ \rangle$ gives $\langle f , f \rangle = \langle f \circ \varphi_r , f \circ \varphi_r \rangle$ for all f in H. Let $f(z) = 1 - r z_1$, then

$$\langle f , f \rangle = \langle 1 , 1 \rangle + r^2 \langle z_1 , z_1 \rangle$$

by (1) and

$$f \circ \varphi_r(z) = 1 - r \frac{r - z_1}{1 - r z_1} = \frac{1 - r z_1 - r^2 + r z_1}{1 - r z_1} = \frac{1 - r^2}{1 - r z_1} .$$

Therefore,

$$\begin{aligned} \langle 1 , 1 \rangle + r^2 \langle z_1 , z_1 \rangle &= \left\langle \frac{1 - r^2}{1 - r z_1} , \frac{1 - r^2}{1 - r z_1} \right\rangle \\ &= (1 - r^2)^2 \sum_{k , j = 0}^{\infty} \langle r^k z_1^k , r^j z_1^j \rangle \\ &= (1 - r^2)^2 \sum_{k=0}^{\infty} r^{2k} \langle z_1^k , z_1^k \rangle \\ &= \langle 1 , 1 \rangle + r^2 \langle z_1 , z_1 \rangle - 2 r^2 \langle 1 , 1 \rangle \\ &\quad + \sum_{k=2}^{\infty} (\langle z_1^k , z_1^k \rangle - 2 \langle z_1^{k-1} , z_1^{k-1} \rangle + \langle z_1^{k-2} , z_1^{k-2} \rangle) r^{2k} . \end{aligned}$$

Since $r \in [0, 1)$ is arbitrary, we must have $\langle 1, 1\rangle = 0$ and

$$\langle z_1^k, z_1^k\rangle - 2\langle z_1^{k-1}, z_1^{k-1}\rangle + \langle z_1^{k-2}, z_1^{k-2}\rangle = 0, \qquad k \geq 2.$$

Now it is easy to show by induction that

$$\langle z_1^k, z_1^k\rangle = k\langle z_1, z_1\rangle$$

for all $k = 0, 1, 2, \dots$, completing the proof of (3).

In order to prove (2), we need the following

Lemma 5. *Suppose $k \geq 0$ is an integer and λ_α $(|\alpha| = k)$ are complex numbers such that*

$$\sum_{|\alpha|=k} \frac{k!}{\alpha_1! \cdots \alpha_n!} t_1^{\alpha_1} \cdots t_n^{\alpha_n} \lambda_\alpha = 1$$

for all $t_1 + \cdots + t_n = 1$ and $t_j > 0$ $(1 \leq j \leq n)$, then $\lambda_\alpha = 1$ for all $|\alpha| = k$.

Proof. The assertion is clearly true for $n = 1$. We may as well assume that $n \geq 2$. since

$$\sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} t_1^{\alpha_1} \cdots t_n^{\alpha_n} = 1$$

for all $t_1 + \cdots + t_n = 1$, we have

$$\sum_{|\alpha|=k} a_\alpha t_1^{\alpha_1} \cdots t_n^{\alpha_n} = 0$$

for all $t_1 + \cdots + t_n = 1$ and $t_j > 0$ $(1 \leq j \leq n)$, where

$$a_\alpha = \frac{|\alpha|!}{\alpha!}(\lambda_\alpha - 1), \qquad |\alpha| = k.$$

For $|\alpha| = k$ and $t_1 + \cdots + t_n = 1$, we can write

$$\begin{aligned}
&\sum_{|\alpha|=k} a_\alpha t_1^{\alpha_1} \cdots t_n^{\alpha_n} \\
&\quad = \sum_{|\alpha|=k} a_\alpha t_1^{\alpha_1} \dots t_{n-1}^{\alpha_{n-1}} (1 - t_1 - \cdots - t_{n-1})^{k-\alpha_1-\cdots-\alpha_{n-1}} \\
&\quad = (1 - t_1 - \cdots - t_{n-1})^k \\
&\qquad \times \sum_{|\alpha|=k} a_\alpha \left(\frac{t_1}{1 - t_1 - \cdots - t_{n-1}}\right)^{\alpha_1} \cdots \left(\frac{t_{n-1}}{1 - t_1 - \cdots - t_{n-1}}\right)^{\alpha_{n-1}}.
\end{aligned}$$

It follows that

$$\sum_{|\alpha|=k} a_\alpha x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} = 0$$

for all $x_j > 0$ $(1 \leq j \leq n-1)$. This implies that $a_\alpha = 0$ $(|\alpha| = k)$. Hence $\lambda_\alpha = 1$ for all $|\alpha| = k$, completing the proof of Lemma 5. □

We can now prove the main equation (2).

Fix any $k \geq 1$ and let $U = (u_{ij})_{n\times n}$ be any unitary matrix. We assume that U acts on $\mathbb{C}^n$ by the usual matrix multiplication Uz, where $z \in \mathbb{C}^n$ is considered a column vector. It follows that

$$z_1 \circ U(z) = u_{11}z_1 + \cdots + u_{1n}z_n .$$

Therefore, the invariance of $\langle\ ,\ \rangle$ implies that

$$\begin{aligned}\langle z_1^k , z_1^k\rangle &= \langle (u_{11}z_1 + \cdots + u_{1n}z_n)^k , (u_{11} , z_1 + \cdots + u_{1n}z_n)^k\rangle \\ &= \sum_{|\alpha|=k}\sum_{|\beta|=k} \frac{|\alpha|!}{\alpha!}\frac{|\beta|!}{\beta!} u_{11}^{\alpha_1}\cdots u_{1n}^{\alpha_n}\bar{u}_{11}^{\beta_1}\cdots \bar{u}_{1n}^{\beta_n}\langle z^\alpha , z^\beta\rangle \\ &= \sum_{|\alpha|=k}\left(\frac{|\alpha|!}{\alpha!}\right)^2 |u_{11}|^{2\alpha_1}\cdots |u_{1n}|^{2\alpha_n}\langle z^\alpha , z^\alpha\rangle .\end{aligned}$$

Since U is unitary, we have $|u_{11}|^2 + \cdots + |u_{1n}|^2 = 1$. Also U is arbitrary, thus we have

$$\langle z_1^k , z_1^k\rangle = \sum_{|\alpha|=k}\left(\frac{|\alpha|!}{\alpha!}\right)^2 t_1^{\alpha_1}\cdots t_n^{\alpha_n}\langle z^\alpha , z^\alpha\rangle$$

for all $t_1 + \cdots + t_n = 1$ and $t_j > 0$ $(1 \leq j \leq n)$. Now if $\langle z_1^k , z_1^k\rangle = 0$, then clearly all $\langle z^\alpha , z^\alpha\rangle = 0$ and so $\langle f , f\rangle = 0$ for all $f \in H$, contradicting the fact that $\langle\ ,\ \rangle$ is nonzero. Thus $\langle z_1^k , z_1^k\rangle \neq 0$ for all $k \geq 1$. Now for any fixed $k \geq 1$, let

$$\lambda_\alpha = \frac{|\alpha|!\langle z^\alpha , z^\alpha\rangle}{\alpha!\langle z_1^k , z_1^k\rangle}, \qquad |\alpha| = k ,$$

then

$$\sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!}t_1^{\alpha_1}\cdots t_n^{\alpha_n}\lambda_\alpha = 1$$

for all $t_1 + \cdots + t_n = 1$ and $t_n > 0$ $(1 \leq j \leq n)$. By Lemma 5, all $\lambda_\alpha = 1$, thus

$$\langle z^\alpha , z^\alpha\rangle = \frac{\alpha!}{|\alpha|!}\langle z_1^{|\alpha|} , z_1^{|\alpha|}\rangle .$$

This completes the proof of Theorem 4. □

4. The existence

By Theorem 4 in the last section, if H is an invariant Hilbert space of holomorphic functions in B_n with semi-inner product $\langle\ ,\ \rangle$, then

$$\langle f , g\rangle = c\sum_\alpha a_\alpha \bar{b}_\alpha \frac{\alpha!}{|\alpha|!}|\alpha|$$

for all $f(z) = \sum_\alpha a_\alpha z^\alpha$ and $g(z) = \sum_\alpha b_\alpha z^\alpha$ in H. The purpose of this section is to show that

$$\langle f , g\rangle = \sum_\alpha a_\alpha \bar{b}_\alpha \frac{\alpha!}{|\alpha|!}|\alpha|$$

is indeed a Möbius invariant inner product.

Theorem 6. *Let H be the space of holomorphic functions $f(z) = \sum_\alpha a_\alpha z^\alpha$ in B_n such that*

$$\sum_\alpha |a_\alpha|^2 \frac{\alpha!}{|\alpha|!}|\alpha| < +\infty .$$

Then H is a Möbius invariant Hilbert space with (semi-) inner product

$$\langle f, g\rangle = \sum_\alpha a_\alpha \bar{b}_\alpha \frac{\alpha!}{|\alpha|!}|\alpha|$$

for all $f(z) = \sum_\alpha a_\alpha z^\alpha$, $g(z) = \sum_\alpha b_\alpha z^\alpha$ in H.

Proof. Clearly H is a nontrivial Hilbert space and the polynomials are dense in H. It only remains to show that its inner product is invariant under $\mathrm{Aut}(B_n)$. By Lemma 1, it suffices to prove the following identities:

$$\langle f\circ U, g\circ U\rangle = \langle f, g\rangle, \tag{1}$$

$$\langle f\circ \varphi_r, g\circ \varphi_r\rangle = \langle f, g\rangle, \tag{2}$$

for all f, g in H and $U \in \mathscr{U}_n$, $r\in [0, 1)$.

In order to prove (1), we recall the homogeneous expansion of holomorphic functions in B_n. Suppose $f(z) = \sum_\alpha a_\alpha z^\alpha$. For any $k \geq 0$, let

$$f_k(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha ,$$

then f_k is a homogeneous polynomial of degree k, and

$$f(z) = \sum_{k=0}^{\infty} f_k(z) .$$

This is called the homogeneous expansion of f. It is easy to see that the homogeneous expansion of f is unique and invariant under linear transformations of $\mathbb{C}^n$. Thus if A is a linear transformation on $\mathbb{C}^n$, then

$$(f\circ A)_k = f_k \circ A .$$

In particular, for all $U\in \mathscr{U}_n$, we have

$$(f\circ U)_k = f_k \circ U .$$

Equation (1) of this section will be a consequence of the following

Lemma 7. *For any $f(z) = \sum_\alpha a_\alpha z^\alpha$, $g(z) = \sum_\alpha b_\alpha z^\alpha$ in H, we have*

$$\sum_\alpha a_\alpha \bar{b}_\alpha \frac{\alpha!}{|\alpha|!}|\alpha| = \sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} k \int_{B_n} f_k(z)\bar{g}_k(z)\, dV(z),$$

where dV is the normalized volume measure on B_n.

Proof. Since the volume measure is rotation invariant, we have

$$\int_{B_n} f_k(z)\bar{g}_k(z)\, dV(z) = \sum_{|\alpha|=k} a_\alpha \bar{b}_\alpha \int_{B_n} |z^\alpha|^2\, dV(z) .$$

By 1.4.9(2) of [7],

$$\int_{B_n} |z^\alpha|^2\, dV(z) = \frac{n!\alpha!}{(n+|\alpha|)!}.$$

Therefore,

$$\int_{B_n} f_k(z)\bar{g}_k(z)\, dV(z) = \sum_{|\alpha|=k} \frac{n!\alpha!}{(n+k)!} a_\alpha \bar{b}_\alpha .$$

It follows that

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} k \int_{B_n} f_k(z)\bar{g}_k(z)\, dV(z) \\
&\quad = \sum_{k=0}^{\infty} \frac{(n+k)!}{n!k!} k \sum_{|\alpha|=k} \frac{n!\alpha!}{(n+k)!} a_\alpha \bar{b}_\alpha \\
&\quad = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{\alpha!}{k!} k a_\alpha \bar{b}_\alpha = \sum_\alpha \frac{\alpha!}{|\alpha|!} |\alpha| a_\alpha \bar{b}_\alpha ,
\end{aligned}$$

and Lemma 7 is proved. □

Now equation (1) of this section follows from the above lemma and the facts that homogeneous expansions are invariant under linear transformations and the volume measure is invariant under unitary transformations of $\mathbb{C}^n$.

In order to prove equation (2) of this section, take $f, g \in H$ and $r \in [0, 1)$. We have

$$\begin{aligned}
\langle f \circ \varphi_r, g \circ \varphi_r \rangle &= \left\langle \sum_\alpha a_\alpha z^\alpha \circ \varphi_r, \sum_\beta b_\beta z^\beta \circ \varphi_r \right\rangle \\
&= \sum_\alpha \sum_\beta a_\alpha \bar{b}_\beta \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle .
\end{aligned}$$

So it suffices to prove

$$\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle = \langle z^\alpha, z^\beta \rangle = \begin{cases} 0, & \alpha \neq \beta, \\ \frac{\alpha!}{|\alpha|!}|\alpha|, & \alpha = \beta. \end{cases}$$

Note that

$$z^\alpha \circ \varphi_r(z) = \left(\frac{r - z_1}{1 - rz_1}\right)^{\alpha_1} \left(-\frac{\sqrt{1-r^2}\, z_2}{1 - rz_1}\right)^{\alpha_2} \cdots \left(-\frac{\sqrt{1-r^2}\, z_n}{1 - rz_1}\right)^{\alpha_n} .$$

When $\alpha_j \neq \beta_j$ for some $2 \leq j \leq n$, then

$$\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle = 0$$

since no monomial in the Taylor expansion of $z^\alpha \circ \varphi_r$ is equal to any monomial in the Taylar expansion of $z^\beta \circ \varphi_r$ (just looking at the powers of z_j). So we may as well assume that

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \qquad \beta = (\beta_1, \alpha_2, \dots, \alpha_n).$$

We will prove in this case that

$$\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle = \begin{cases} 0, & \alpha_1 \neq \beta_1, \\ \langle z^\alpha, z^\alpha \rangle, & \alpha_1 = \beta_1. \end{cases}$$

Let $N = \alpha_2 + \cdots + \alpha_n$, then

$$z^\alpha \circ \varphi_r(z) = (-1)^N (1-r^2)^{\frac{N}{2}} \frac{(r-z_1)^{\alpha_1}}{(1-rz_1)^{\alpha_1+N}} z_2^{\alpha_2} \cdots z_n^{\alpha_n},$$

$$z^\beta \circ \varphi_r(z) = (-1)^N (1-r^2)^{\frac{N}{2}} \frac{(r-z_1)^{\beta_1}}{(1-rz_1)^{\beta_1+N}} z_2^{\alpha_2} \cdots z_n^{\alpha_n}.$$

Let

$$F(z_1) = \frac{(r-z_1)^{\alpha_1}}{(1-rz_1)^{\alpha_1+N}} = \sum_{k=0}^\infty c_k z_1^k,$$

$$G(z_1) = \frac{(r-z_1)^{\beta_1}}{(1-rz_1)^{\beta_1+N}} = \sum_{k=0}^\infty d_k z_1^k,$$

then

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle &= (1-r^2)^N \sum_{k=0}^\infty c_k \bar{d}_k \langle z_1^k z_2^{\alpha_2} \cdots z_n^{\alpha_n}, z_1^k z_2^{\alpha_2} \cdots z_n^{\alpha_n} \rangle \\ &= (1-r^2)^N \sum_{k=0}^\infty c_k \bar{d}_k \frac{k!\alpha_2! \cdots \alpha_n!}{(k+N)!}(k+N). \end{aligned}$$

We first settle the cases $N = 0, 1$.

When $N = 0$, we have

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle &= \sum_{k=0}^\infty k c_k \bar{d}_k = \int_{\mathbb{D}} F'(z_1)\overline{G'(z_1)}\, dA(z_1) \\ &= \int_{\mathbb{D}} \left[\left(\frac{r-z_1}{1-rz_1}\right)^{\alpha_1}\right]' \overline{\left[\left(\frac{r-z_1}{1-rz_1}\right)^{\beta_1}\right]'}\, dA(z_1). \end{aligned}$$

A change of variable $z_1 \to \frac{r-z_1}{1-rz_1}$ leads to

$$\begin{aligned} \langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r \rangle &= \int_{\mathbb{D}} (z_1^{\alpha_1})' \overline{(z_1^{\beta_1})'}\, dA(z_1) \\ &= \begin{cases} 0 & \text{if } \alpha_1 \neq \beta_1, \\ \alpha_1 & \text{if } \alpha_1 = \beta_1, \end{cases} \\ &= \langle z^\alpha, z^\beta \rangle \end{aligned}$$

for $\alpha_2 = \cdots = \alpha_n = 0$. This is actually the one-dimensional result. See [1].

When $N = 1$, we have

$$\begin{aligned}\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r\rangle &= (1-r^2)\sum_{k=0}^\infty c_k \bar{d}_k \\ &= (1-r^2)\frac{1}{2\pi}\int_0^{2\pi} F(e^{i\theta})\overline{G(e^{i\theta})}\,d\theta \\ &= (1-r^2)\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r-e^{i\theta}}{1-re^{i\theta}}\right)^{\alpha_1}\left(\frac{r-e^{-i\theta}}{1-re^{-i\theta}}\right)^{\beta_1} d\theta.\end{aligned}$$

By the change of variable $\frac{r-e^{i\theta}}{1-re^{i\theta}} \to e^{it}$, we get

$$\begin{aligned}\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos t}e^{i(\alpha_1-\beta_1)t}\,dt \\ &= \begin{cases} 0 & \text{if } \alpha_1 \neq \beta_1, \\ 1 & \text{if } \alpha_1 = \beta_1, \end{cases} \\ &= \langle z^\alpha, z^\beta\rangle\end{aligned}$$

for $\alpha_2 + \cdots + \alpha_n = 1$.

Now if $N \geq 2$, then

$$\begin{aligned}\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r\rangle &= (1-r^2)^N\sum_{k=0}^\infty c_k \bar{d}_k \frac{k!\alpha_2!\cdots\alpha_n!}{(k+N)!}(k+N) \\ &= (1-r^2)^N\frac{\alpha_2!\cdots\alpha_n!}{(N-1)!}\sum_{k=0}^\infty c_k \bar{d}_k\frac{k!(N-1)!}{(k+N-1)!}.\end{aligned}$$

By 1.4.9(2) of [7],

$$\frac{k!(N-1)!}{(k+N-1)!} = \int_{B_{N-1}} |z_1^k|^2\,dV(z),$$

where dV is the normalized volume measure on B_{N-1}. Therefore,

$$\begin{aligned}\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r\rangle &= (1-r^2)^N\frac{\alpha_2!\cdots\alpha_n!}{(N-1)!}\sum_{k=0}^\infty c_k \bar{d}_k\int_{B_{N-1}}|z_1^k|^2\,dV(z) \\ &= (1-r^2)^N\frac{\alpha_2!\cdots\alpha_n!}{(N-1)!}\int_{B_{n-1}} F(z_1)\overline{G(z_1)}\,dV(z).\end{aligned}$$

By Lemma 3,

$$\int_{B_{N-1}} F(z_1)\overline{G(z_1)}\,dV(z) = (N-1)\int_{\mathbb{D}} F(z_1)\overline{G(z_1)}(1-|z_1|^2)^{N-2}\,dA(z_1).$$

Thus

$$\begin{aligned}\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r\rangle = \frac{\alpha_2!\cdots\alpha_n!}{(N-2)!}\int_{\mathbb{D}}\left(\frac{r-z_1}{1-rz_1}\right)^{\alpha_1}\overline{\left(\frac{r-z_1}{1-rz_1}\right)^{\beta_1}} \\ \times\frac{(1-r^2)^N(1-|z_1|^2)^N}{|1-rz_1|^{2N}}\frac{dA(z_1)}{(1-|z_1|^2)^2}.\end{aligned}$$

Note that

$$\frac{(1-r^2)^N(1-|z_1|^2)^N}{|1-rz_1|^{2N}} = \left(1-\left|\frac{r-z_1}{1-rz_1}\right|^2\right)^N$$

and the measure $dA(z_1)/(1-|z_1|^2)^2$ is invariant under Möbius transformations of $\mathbb{D}$, thus a change of variable gives

$$\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r\rangle = \frac{\alpha_2! \cdots \alpha_n!}{(N-2)!} \int_{\mathbb{D}} z_1^{\alpha_1} \bar{z}_1^{\beta_1} (1-|z_1|^2)^N \frac{dA(z_1)}{(1-|z_1|^2)^2}.$$

Since the measure $(1-|z_1|^2)^{N-2}\, dA(z_1)$ is rotation invairant, we have

$$\int_{\mathbb{D}} z_1^{\alpha_1} \overline{z_1^{\beta_1}} (1-|z_1|^2)^{N-2}\, dA(z_1) = 0$$

if $\alpha_1 \neq \beta_1$. Thus

$$\langle z^\alpha \circ \varphi_r, z^\beta \circ \varphi_r\rangle = \langle z^\alpha, z^\beta\rangle$$

if $\alpha \neq \beta$. If $\alpha_1 = \beta_1$, then $\alpha = \beta$ and

$$\begin{aligned}\langle z^\alpha \circ \varphi_r, z^\alpha \circ \varphi_r\rangle &= \frac{\alpha_2! \cdots \alpha_n!}{(N-2)!} \int_{\mathbb{D}} |z_1|^{2\alpha_1} (1-|z_1|^2)^{N-2}\, dA(z_1)\\ &= \frac{\alpha_2! \cdots \alpha_n!}{(N-2)!} \int_0^1 t^{\alpha_1}(1-t)^{N-2}\, dt.\end{aligned}$$

Using the classical B-function and Γ-function, we have

$$\int_0^1 t^{\alpha_1}(1-t)^{N-2}\, dt = B(\alpha_1+1, N-1) = \frac{\Gamma(\alpha_1+1)\Gamma(N-1)}{\Gamma(\alpha_1+N)} = \frac{\alpha_1!(N-2)!}{(\alpha_1+N-1)!}.$$

Therefore,

$$\langle z^\alpha \circ \varphi_r, z^\alpha \circ \varphi_r\rangle = \frac{\alpha_1! \cdots \alpha_n!}{(\alpha_1+N-1)!} = \frac{\alpha!}{|\alpha|!}|\alpha| = \langle z^\alpha, z^\alpha\rangle.$$

This completes the rpoof of Theorem 6. □

5. Other descriptions of the invariant inner product

The results in this section are somewhat negative. We point out many other ways of constructing the invariant inner product on the Dirichlet space $\mathscr{D}$ of the open unit disc $\mathbb{D}$. Then we show one-by-one that these constructions fail in higher dimensions.

The Dirichlet pairing

$$\langle f, g\rangle_{\mathscr{D}} = \int_{\mathbb{D}} f'(z)\overline{g'(z)}\, dA(z)$$

is clearly an invariant inner product on the Dirichlet space $\mathscr{D}$. Our problem here is to try to find a natural analog of this inner product in higher dimensions.

As the first trial, one may be tempted to look at the pairing

$$\langle f, g\rangle = \int_{B_n} \langle \nabla f(z), \nabla g(z)\rangle_{\mathbb{C}^n}\, dV(z),$$

where $\nabla f(z)$ is the complex gradient of f at z. This is indeed an inner product, but it is easy to see that it is not invariant when $n \geq 2$.

Recall that the invariant Laplacian $\widetilde{\Delta}$ of B_n is defined by

$$\widetilde{\Delta} f(z) = \Delta(f \circ \varphi_z)(0),$$

where Δ is the usual Laplacian in $\mathbb{C}^n$. $\widetilde{\Delta}$ is invariant in the sense that

$$\widetilde{\Delta}(f \circ \varphi)(z) = (\widetilde{\Delta} f) \circ \varphi(z)$$

for all $\varphi \in \mathrm{Aut}(B_n)$. On the unit disc $\mathbb{D}$,

$$\widetilde{\Delta}(|f|^2)(z) = 4(1 - |z|^2)^2 |f'(z)|^2$$

for all holomorphic functions f. It follows that

$$\langle f, g \rangle_{\mathscr{D}} = \int_{\mathbb{D}} \widetilde{\Delta}(f\bar{g})(z) K(z, z)\, dA(z),$$

where $K(z, w)$ is Bergman kernel of $\mathbb{D}$:

$$K(z, w) = \frac{1}{(1 - z\bar{w})^2}.$$

The above inner product is invariant because $\widetilde{\Delta}$ is invariant and the Bergman kernel is invariant. Naturally we look at the generalization to B_n:

$$\langle f, g \rangle = \int_{B_n} \widetilde{\Delta}(f\bar{g})(z) K(z, z)\, dV(z),$$

where $K(z, w)$ is the Bergman kernel of B_n:

$$K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1}}.$$

The pairing is clearly Möbius invariant since both $\widetilde{\Delta}$ and the Bergman kernel are invairant. Unfortunately, when $n \geq 2$, the only holomorphic functions f in B_n with

$$\int_{B_n} \widetilde{\Delta}(|f|^2)(z) K(z, z)\, dV(z) < +\infty$$

are the constant functions. See [8].

Let P be the Bergman projection defined by

$$Pf(z) = \int_{B_n} K(z, w) f(w)\, dV(w).$$

Given a function $g \in L^2(B_n, dV)$, the Hankel operator H_g on the Bergman space

$$L_a^2(B_n) = \{f \in L^2(B_n, dV) : f \text{ is holomorphic}\}$$

is defined by

$$H_g f = (I - P)(fg),$$

where I is the identity operator on $L^2(B_n, dV)$. Basic properties of Hankel operators can be found in [3–5]. Hankel operators depend on its symbol invariantly, that is, if $\varphi \in \mathrm{Aut}(B_n)$, then

$$H_{g\circ\varphi} = U_\varphi H_g U_\varphi^*,$$

where U_φ is the unitary operator on $L^2(B_n, dV)$ defined by

$$U_\varphi f(z) = J_\varphi(z) f(\varphi(z)),$$

$J_\varphi(z)$ is the complex Jacobian determinant of φ at z. It was shown in [3] that if f is holomorphic in $\mathbb{D}$, then $H_{\bar f}$ is Hilbert-Schmidt iff $f \in \mathscr{D}$. Moreover,

$$\langle f, g\rangle_{\mathscr{D}} = \mathrm{tr}(H_{\bar f}^* H_{\bar g})$$

for all f, g in $\mathscr{D}$, where tr denotes the trace of an operator. Once again, this nice result does not generalize to higher dimensions. As shown in [8], when $n \geq 2$, the only Hilbert-Schmidt Hankel operator $H_{\bar f}$ with f holomorphic in B_n is the zero operator (when f is a constant).

The so-called Berezin transform is an invariant transform which has attracted much attention lately in function theory and operator theory [4, 5]. We have a brief discussion of it here because of its connection with invariant inner products.

Given a function f in $L^1(B_n, dV)$, let

$$\tilde{f}(z) = \int_{B_n} f(w) \frac{|K(z, w)|^2}{K(z, z)} \, dV(w), \qquad z \in B_n.$$

The function $\tilde f$ is called the Berezin transform of f (see [4, 5]). Clearly, $\tilde{\bar f} = \bar{\tilde f}$, and the reproducing property of $K(z, w)$ gives $\tilde f = f$ if f is holomorphic. The invariance of the Bergman kernel $K(z, w)$ implies the invariance of the Berezin transform namely,

$$\widetilde{f \circ \varphi}(z) = \tilde f(\varphi(z))$$

for all $\varphi \in \mathrm{Aut}(B_n)$.

Now consider the pairing

$$\langle f, g\rangle = \int_{B_n} (\widetilde{f\bar g}(z) - \tilde f(z)\tilde{\bar g}(z)) K(z, z) \, dV(z)$$

and the space

$$H_n = \left\{ f\colon \int_{B_n} (\widetilde{|f|^2}(z) - |\tilde f(z)|^2) K(z, z)\, dV(z) < +\infty, \quad f \text{ holomorphic in } B_n \right\}.$$

Since the Berezin transform is invariant and the measure $K(z, z)\, dV(z)$ is also invariant, H_n is an invariant space. It is also easy to see that $\mathscr{U}_n$ acts on H_n

continuously, thus by Lemma 2, H_n will be an invariant Hilbert space if H_n contains a nonconstant function.

When $n = 1$, we show that the function $f(z) = z$ is in H_1. In fact, for $f(z) = z$ in $\mathbb{D}$, we have

$$\widetilde{|f|^2}(z) - |\widetilde{f}(z)|^2 = \int_{\mathbb{D}} |f \circ \varphi_z(w) - f(z)|^2 \, dA(w) \quad \text{(see [4])}$$
$$= \int_{\mathbb{D}} \left| \frac{z - w}{1 - \bar{z}w} - z \right|^2 dA(w)$$
$$= \frac{(1 - |z|^2)^2}{|z|^4} \left[\log \frac{1}{1 - |z|^2} - |z|^2 \right].$$

It follows that for $f(z) = z$ in $\mathbb{D}$,

$$\int_{\mathbb{D}} (\widetilde{|f|^2}(z) - |\widetilde{f}(z)|^2) K(z, z) \, dA(z) = \int_{\mathbb{D}} \frac{1}{|z|^4} \left[\log \frac{1}{1 - |z|^2} - |z|^2 \right] dA(z) < +\infty.$$

By the uniqueness of invariant Hilbert spaces, we must have $\mathscr{D} = H_1$ and

$$\langle f, g \rangle_{\mathscr{D}} = c^{-1} \int_{\mathbb{D}} (\widetilde{f\bar{g}}(z) - \widetilde{f}(z)\widetilde{\bar{g}}(z)) K(z, z) \, dA(z),$$

where

$$c = \int_{\mathbb{D}} \frac{1}{|z|^4} \left[\log \frac{1}{1 - |z|^2} - |z|^2 \right] dA(z) = 1.$$

Once again this construction fails when $n \geq 2$. The proof is essentially contained in [8].

Finally in this section, we describe a general method of constructing the invariant inner product on the Dirichlet space $\mathscr{D}$.

Suppose F is a positive continuous function defined on $[0, +\infty)$. Consider the pairing

$$\langle f, g \rangle_F = \int_{B_n} \int_{B_n} (f(z) - f(w))(\bar{g}(z) - \bar{g}(w)) |K(z, w)|^2 F(|\varphi_z(w)|) \, dV(z) \, dV(w).$$

Let $\rho(z, w) = |\varphi_z(w)|$ be the pseudohyperbolic distance on B_n. ρ is invariant in the sense that

$$\rho(\varphi(z), \varphi(w)) = \rho(z, w), \qquad \varphi \in \mathrm{Aut}(B_n).$$

The invariance of $K(z, w)$ and ρ implies that $\langle \ , \ \rangle_F$ is an invariant pairing; that is,

$$\langle f \circ \varphi, g \circ \varphi \rangle_F = \langle f, g \rangle_F$$

for all f and g in the space

$$H_F = \{ f : \langle f, f \rangle_F < +\infty, \ f \text{ holomorphic} \}.$$

It is easy to check that $\mathscr{U}_n$ acts on H_F continuously. Thus by Lemma 2, H_F is an invariant Hilbert space of holomorphic functions in B_n iff H_F contains a nonconstant function.

Proposition 8. *When* $n = 1$, *we have* $\mathscr{D} = H_F$ *iff* $\int_0^1 F(\sqrt{r}) \log \frac{1}{1-r}\, dr < +\infty$.

Proof. By the above remarks, we have $\mathscr{D} = H_F$ iff $z \in H_F$. First observe that

$$\begin{aligned}\langle f, f\rangle_F &= \int_{B_n}\int_{B_n} |f(z) - f(w)|^2 |K(z, w)|^2 F(|\varphi_z(w)|)\, dV(z)\, dV(w) \\ &= \int_{B_n} K(z, z)\, dV(z) \int_{B_n} |f(z) - f(w)|^2 F(|\varphi_z(w)|) \frac{|K(z, w)|^2}{K(z, z)}\, dV(w).\end{aligned}$$

Since the real Jacobian determinent of φ_z at w is precisely $|K(z, w)|^2/K(z, z)$ (see 2.2.6 of [7]), a change of variable now gives

$$\begin{aligned}\langle f, f\rangle_F &= \int_{B_n} K(z, z)\, dV(z) \int_{B_n} |f(z) - f \circ \varphi_z(w)|^2 F(|w|)\, dV(w) \\ &= \int_{B_n} F(|w|)\, dV(w) \int_{B_n} |f(z) - f \circ \varphi_z(w)|^2 K(z, z)\, dV(z).\end{aligned}$$

When $n = 1$ and $f(z) = z$, we have

$$f(z) - f \circ \varphi_z(w) = \frac{w(1 - |z|^2)}{1 - \bar{z}w},$$

thus

$$\int_{\mathbb{D}} |f(z) - f \circ \varphi_z(w)|^2 K(z, z)\, dA(z) = \int_{\mathbb{D}} \frac{|w|^2}{|1 - \bar{z}w|^2}\, dA(z) = \log \frac{1}{1 - |w|^2}.$$

Therefore,

$$\begin{aligned}\langle z, z\rangle_F &= \int_{\mathbb{D}} F(|w|) \log \frac{1}{1 - |w|^2}\, dA(w) \\ &= 2\int_0^1 rF(r) \log \frac{1}{1 - r^2}\, dr \\ &= \int_0^1 F(\sqrt{r}) \log \frac{1}{1 - r}\, dr.\end{aligned}$$

Hence $\mathscr{D} = H_F$ iff $z \in H_F$ iff $\int_0^1 F(\sqrt{r}) \log \frac{1}{1-r}\, dr < +\infty$. This completes the proof of Proposition 8. □

Corollary 9. *For any* f *holomorphic in* $\mathbb{D}$ *and* $F\colon [0, \infty) \to (0, +\infty)$ *continuous, we have*

$$\begin{aligned}&\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\bar{w}|^4} F\left(\left|\frac{z - w}{1 - z\bar{w}}\right|\right) dA(z)\, dA(w) \\ &\quad = \int_{\mathbb{D}} |f'(z)|^2\, dA(z) \cdot \int_0^1 F(\sqrt{r}) \log \frac{1}{1 - r}\, dr.\end{aligned}$$

Note that when $F \equiv 1$, we have

$$\int_0^1 \log \frac{1}{1 - r}\, dr = 1$$

and so

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|f(z)-f(w)|^2}{|1-z\bar{w}|^4}\, dA(z)\, dA(w) = \int_{\mathbb{D}} |f'(z)|^2\, dA(z)$$

for any holomorphic function f in $\mathbb{D}$. By choosing other special functions f, we can obtain many other interesting integral formulae.

We remark that the above construction on $\mathbb{D}$ does not generalize to B_n when $n \geq 2$. Namely, when $n \geq 2$, the only way to make

$$\int_{B_n}\int_{B_n} |f(z)-f(w)|^2 |K(z, w)|^2 F(|\varphi_z(w)|)\, dV(z)\, dV(w) < +\infty$$

is to choose f to be a constant or F identically zero. The argument is elementary and it reduces to showing that

$$\int_{B)n} |z - \varphi_z(w)|^2 K(z, z) dV(z) = +\infty$$

for all $n \geq 2$ and $w \neq 0$. The details are omitted here.

6. Other descriptions of the invariant Hilbert space

Although we have been unsuccessful in finding a simple intrinsic description of the invariant inner product, there is a way of describing the invariant Hilbert space in terms of the (generalized) radial derivative. What follows is essentially suggested by the referee. The author thanks the referee for his useful comments.

Suppose $f(z)$ is holomorphic in B_n. The radial derivative of f is

$$Rf(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z).$$

If $f(z) = \sum_\alpha a_\alpha z^\alpha$, then it is easy to see that

$$Rf(z) = \sum_\alpha |\alpha| a_\alpha z^\alpha .$$

Given any real number t, define an operator R^t on the space of holomorphic functions in B_n as follows:

$$R^t\left(\sum_\alpha a_\alpha z^\alpha\right) = \sum_{\alpha\neq 0} |\alpha|^t a_\alpha z^\alpha .$$

We can think of these operators as generalized radial derivatives.

Theorem 10. *Suppose $t > \frac{n}{2}$ and $f(z) = \sum_\alpha a_\alpha z^\alpha$ is holomorphic in B_n, then*

$$\sum_\alpha |a_\alpha|^2 \frac{\alpha!}{|\alpha|!}\, |\alpha| < +\infty$$

if and only if

$$(1-|z|^2)^t R^t f(z) \in L^2(B_n, d\lambda),$$

where

$$d\lambda(z) = K(z, z)\,dV(z) = \frac{dV(z)}{(1-|z|^2)^{n+1}}$$

is the Möbius invariant measure on B_n.

Proof. Fix $f(z) = \sum_\alpha a_\alpha z^\alpha$ holomorphic in B_n. Since $R^t f(z) = \sum |\alpha|^t a_\alpha z^\alpha$ and $(1-|z|^2)^{2t}\,d\lambda(z)$ is invariant under unitary transformations of $\mathbb{C}^n$, we have

$$\begin{aligned}
&\int_{B_n} (1-|z|^2)^{2t} |R^t f(z)|^2\, d\lambda(z) \\
&\quad = \sum_\alpha |\alpha|^{2t} |a_\alpha|^2 \int_{B_n} (1-|z|^2)^{2t} |z^\alpha|^2\, d\lambda(z) \\
&\quad = \sum_\alpha |\alpha|^{2t} |a_\alpha|^2 \int_{B_n} (1-|z|^2)^{2t-(n+1)} |z^\alpha|^2\, dV(z).
\end{aligned}$$

Using polar coordinates on B_n (1.4.3 of [7]), we get

$$\begin{aligned}
&\int_{B_n} (1-|z|^2)^{2t-(n+1)} |z^\alpha|^2\, dV(z) \\
&\quad = 2n \int_0^1 r^{2n-1} (1-r^2)^{2t-(n+1)} r^{2|\alpha|}\, dr \int_{\partial B_n} |w^\alpha|^2\, d\sigma(w),
\end{aligned}$$

where $d\sigma$ is the normalized area measure on ∂B_n. By 1.4.9 of [7],

$$\int_{\partial B_n} |w^\alpha|^2\, d\sigma(w) = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}.$$

Using Γ-functions, we have

$$\begin{aligned}
&2\int_0^1 r^{2n-1}(1-r^2)^{2t-(n+1)} r^{2|\alpha|}\, dr \\
&\quad = \int_0^1 t^{n-1}(1-r)^{2t-(n+1)} r^{|\alpha|}\, dr \\
&\quad = \int_0^1 r^{n+|\alpha|-1}(1-r)^{2t-n-1}\, dr \\
&\quad = \frac{\Gamma(n+|\alpha|)\Gamma(2t-n)}{\gamma(|\alpha|+2t)} = \frac{(n+|\alpha|-1)!\Gamma(2t-n)}{\Gamma(|\alpha|+2t)}.
\end{aligned}$$

It follows that

$$\int_{B_n} (1-|z|^2)^{2t-(n+1)} |z^\alpha|^2\, dV(z) = \frac{n!\alpha!\Gamma(2t-n)}{\Gamma(|\alpha|+2t)}$$

and

$$\int_{B_n} (1-|z|^2)^{2t} |R^t f(z)|^2 \, d\lambda(z)$$
$$= n!\Gamma(2t-n) \sum_{\alpha} |a_\alpha|^2 |\alpha|^{2t} \frac{\alpha!}{\Gamma(|\alpha|+2t)}$$
$$= n!\Gamma(2t-n) \sum_{\alpha} |a_\alpha|^2 \frac{\alpha!}{|\alpha|!} |\alpha| \frac{|\alpha|!|\alpha|^{2t-1}}{\Gamma(|\alpha|+2t)} .$$

Stirling's formula shows that

$$\lim_{|\alpha|\to+\infty} \frac{|\alpha|!|\alpha|^{2t-1}}{\Gamma(|\alpha|+2t)} = 1 .$$

Thus

$$\int_{B_n} (1-|z|^2)^{2t} |R^t f(z)|^2 \, d\lambda(z) < +\infty$$

if and only if

$$\sum_{\alpha} |a_\alpha|^2 \frac{\alpha!}{|\alpha|!} |\alpha| < +\infty ,$$

completing the proof of Theorem 10. □

Corollary 11. *If* $f(z) = \sum_\alpha a_\alpha z^\alpha$ *is holomorphic in* B_n, *then* $\sum_\alpha |a_\alpha|^2 \frac{\alpha!}{|\alpha|!} |\alpha| < +\infty$ *if and only if*

$$\int_{B_n} |R^{(n+1)/2} f(z)|^2 \, dV(z) < +\infty .$$

Finally we point out that it is also possible to characterize the invariant Hilbert space in terms of the (generalized) radial derivative on the boundary of B_n. In fact, the referee computed that for $f(z) = \sum_\alpha a_\alpha z^\alpha$ holomorphic in B_n, $\sum_\alpha |a_\alpha|^2 \frac{\alpha!}{|\alpha|!} |\alpha| < +\infty$ if and only if

$$\int_{\partial B_n} |R^{n/2} f(w)|^2 \, d\sigma(w) < +\infty .$$

The proof is similar to that of Theorem 10, we omit the details here.

Note added in proof. After the present paper was accepted for publication, the author was informed that Jaak Peetre obtained essentially the same results (uniqueness and existence) in 1984 in an unpublished manuscript. However, Peetre's method was quite different from the computational approach here.

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Department of Mathematics and Statistics, SUNY at Albany, 1400 Washington Avenue, Albany, New York 12222